

“Relating Missing and Decycling Edges in Directed Graphs”

Sean Karson, Trinity Preparatory School, Winter Park, FL, Kevin Chen, William P. Clements High School, Sugar Land, TX and Dan Liu, Liberal Arts and Science Academy High School, Austin, TX – 2009 National Team Winners

Abstract: The Chudnovsky-Seymour-Sullivan (CSS) conjecture, proposed in 2006, delves into the relationship between two functions on any simple, triangle-free digraph G , namely $\beta(G)$ and $\gamma(G)$. $\beta(G)$ denotes the minimum number of edges that must be removed to eliminate all directed cycles found in G , and $\gamma(G)$ is denoted to be the number of unordered pairs of vertices without an edge between them, or simply the number of non-edges. In their paper Chudnovsky, Seymour, and Sullivan proved that for any simple, triangle-free digraph, $\beta(G) \leq 1.0\gamma(G)$ and conjectured that $\beta(G) \leq 0.5\gamma(G)$. Recently, Dunkum, Hamburger, and Por made some headway, showing that $\beta(G) \leq 0.88\gamma(G)$. In this paper we lower the bound on $\beta(G)$ even further, by proving that $\beta(G) \leq 0.8616\gamma(G)$. Separately, we prove that $\beta(G) \leq 0.5\gamma(G)$ for two subcases that include planar and bipartite digraphs and then offer ideas for methods with which one could achieve better results relating to the CSS conjecture. We describe how our main result has already been used in a related graph theory paper and finally demonstrate that our work has broad implications in the field of graph theory because of recent connections made between the CSS conjecture and the still open, decades-old Caccetta-Haggkvist conjecture, a problem whose solution promises improved network efficiency in a wide variety of applications.

Mentor: Dr. Jian Shen

“Matching Preclusions for Augmented Cubes”

Randy Jia and David Lu, Detroit Country Day School, Beverly Hills, MI – 2009 National Team Finalists

Abstract: This study examines the matching preclusion number as it relates to the augmented cube graph. The matching preclusion number is the minimum number of edges to delete in a graph such that no perfect matching remains. A perfect matching of a graph is a set of edges such that every vertex is incident to exactly one edge. This topic is of importance in a variety of issues relating to network connectivity. By examining matching preclusion numbers of different graphs, the relative connectivity of different graphs can be compared, and the classification of such matching preclusion sets makes it possible to relate different graphs to others. We explore the matching preclusion number as it relates to the graph of the augmented cube. We determine the matching preclusion number for the augmented cube and show that the only optimal matching preclusion involves isolating a single vertex. Moreover, we consider and classify conditional matching preclusion sets, which have recently been introduced as preclusion sets which do not allow for isolated vertices.

Mentor: Dr. Eddie Cheng

“Chomp on Graphs and Subsets”

Lynnelle Ye, Palo Alto Senior High School, Palo Alto, CA – 2009 National Individual Finalist

Abstract: The game subset take-away begins with a simplicial complex Δ . Two players take turns removing any element of Δ as well as all other elements which contain it, and the last player able to move wins. Graph chomp is a special case of subset take-away played on a simplicial complex with only vertices and edges. The game has previously only been analyzed for complete graphs, forest graphs, and very small special cases of higher-dimensional simplicial complexes. We generalize a common method of reducing some game positions to simpler ones by symmetry and provide a complete analysis of complete n -partite graphs for arbitrary n and all bipartite graphs. Finally, we give partial results for odd-cycle pseudotrees, which are non-bipartite graphs with a single cycle.

Mentor: Mr. Tirasan Khandhawit

“Probabilistic Properties of the Logarithmic Number Derivative”

Tim Kunisky, Livingston High School, Livingston, NJ – 2009 National Individual Finalist

Abstract: The subject of this paper is the logarithmic number derivative, a completely additive arithmetic function defined on the prime numbers by $L(p) = p^{-1}$. We derive an improved expression for the consecutive sum of $L(n)$ and by corollary find its average order, and prove the existence of and bounds on the average order of $L(n)^2$. Then, we show the existence of a pure, continuous limiting distribution of finite mean and variance for $L(n)$, and find an Euler product for its characteristic function $\varphi(\tau)$. $L(z, n)$ is defined as a multiplicative analogue of $L(n)$, and we investigate expressions for its average order and Dirichlet series. An application of the Selberg-Delange method to this Dirichlet series yields that $\varphi(\tau)$ differs from the mean value over $n \leq x$ of $L(e^{it}, n)$ by a term of magnitude $O(\exp(-c \log(x)^{1/2}))$ for a positive constant c . We then consider the limiting distribution of $L(n) \bmod 1$, which is shown to be existent and continuous, but not absolutely continuous, which implies that it must have pathological properties. Finally, we propose a generalization of the $L(n)$ and $\Omega(n)$ functions, and discuss several questions open for future investigation regarding both $L(n)$ and this generalization.

Mentor: Dr. Alex Kontorovich

“Previously Unknown Parts of the Greene-Kleitman Partition for the Tamari Lattice”

Raphael-Joel Lim, The Indiana Academy for Science, Math, and Humanities, Muncie, IN and Mark Zhang, William P. Clements High School, Sugar Land, TX – 2008 National Team Finalists

Abstract: For every partially-ordered set P there exists a partition $\lambda(P)$ such that the sum of the first k parts of $\lambda(P)$ is the size of the largest union of k chains in P . For a certain well-known partially-ordered set, the Tamari lattice T_n , the first three values of $\lambda(T_n)$ have been determined for n sufficiently large.

In this paper, we prove that the fourth and fifth values of $\lambda(T_n)$ are both two less than the third value. To do this, we first find upper bounds for these values by proving the existence of a very restrictive partition of T_n into antichains, utilizing more elegant methods than any previously implemented. Then, we explicitly define chains which achieve those upper bounds. The strategies employed here can be extended to determine more values of $\lambda(T_n)$ and reduce what was for thirty years a daunting open problem to a task greatly more tractable.

Mentor: Dr. Edward Early

“The Classification of Certain Fusion Categories”

Eric Larson, South Eugene High School, Eugene, OR – 2008 National Individual Finalist

Abstract: Fusion categories are algebraic structures that generalize finite groups, semisimple Hopf and quasi-Hopf algebras. They arise and have applications in various areas of mathematics, as well as theoretical physics and computer science, such as representation theory, low-dimensional topology, theory of operator algebras, string theory, conformal field theory and quantum computing. Classification of fusion categories of a given dimension is an important problem which extends the classical problem of classification of finite groups of a fixed order. This problem is very hard and was solved only for dimensions equal to p^n , pq or pqr , where p , q and r are distinct primes. In this paper we advance this classification program in two directions. We completely classify fusion categories of dimension pq^2 , where p and q are distinct primes. This case is especially interesting because it is the first class of integral fusion categories where not all members are group-theoretical. Our second theorem is a classification of a certain family of $\mathbb{Z}/3\mathbb{Z}$ -graded fusion categories. It is a generalization of the 1998 result of Tambara and Yamagami. The proofs of our results are based on the recently developed theory of extensions of fusion categories.

Mentor: Dr. Pavel Etingof

“Associated Primes of the Square of the Alexander Dual of Hypergraphs”

Ashok Cutkosky, David H. Hickman High School, Columbia, MO – 2008 National Individual Finalist

Abstract: The purpose of this paper is to provide methods for determining the associated primes of $(I(H))^2$ for an m -hypergraph H , with a specific regard towards 3-hypergraphs. We prove a general method for detecting associated primes of the square of the Alexander dual of the edge ideal based on combinatorial conditions on the m -hypergraph. Also, we demonstrate a more efficient combinatorial criterion for detecting the non-existence of non-minimal associated primes. In investigating 3-hypergraphs, we prove a surprising generalization of the previously discovered results for 2-hypergraphs (simple graphs). For 2-hypergraphs, associated primes of the square of the Alexander dual of the edge ideal are either of height 2 or of odd height greater than 2. However, we prove that in the 3-hypergraph case, there is no such restriction – or indeed any restriction – on the heights of the associated primes. Specifically, given any integers q and n with $3 \leq q \leq n$, we construct a 3-hypergraph of size n with an associated prime of height q . We further prove that it is possible to construct connected 3-hypergraphs under the same conditions.

Mentor: Dr. Christopher Francisco

“Cayley graphs formed by conjugate generating sets of S_n ”

Jacob Steinhardt, Thomas Jefferson High School for Science and Technology, Alexandria, VA – 2007 National Individual Finalist

Abstract: We investigate subsets of the symmetric group with structure similar to that of a graph. The “trees” of these subsets correspond to minimal conjugate generating sets of the symmetric group. We prove two main theorems. The first is a characterization of minimal conjugate generating sets of S_n . The second is a generalization of a result due to Feng characterizing the automorphism groups of the Cayley graphs formed by certain generating sets composed of cycles. We compute the full automorphism groups subject to a weak condition and conjecture that the characterization still holds without the condition. We also present some computational results in relation to the Lovasz conjecture, including a generalization of the work on quasi-hamiltonicity by Gutin and Yeo to undirected graphs.

Mentor: N/A